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4501. In this question, do not use a calculator.

A quartic equation is given as

$$16x^4 - 32x^3 + 32x^2 - 16x - 21 = 0.$$

Using the substitution $z = 2x - 1$, or otherwise, determine the solution of this equation.

4502. The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}.$$

(a) Prove that, if the iteration has a fixed point, then that fixed point must be a root of the equation $\mathrm{f}(x) = 0$.

(b) Show algebraically that, when the iteration is applied to the equation $x^k = 0$, for $k > 1$, the iteration will converge to $x = 0$, irrespective of the starting value.

4503. You are given that the gradient of a curve, which passes through the origin, satisfies

$$\sqrt{y}\,\mathrm{cosec}^2\,x \frac{dy}{dx} = \tfrac{1}{3}.$$

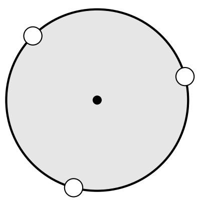
Show that $y = \frac{1}{4}\left(2x - \sin 2x\right)^{\frac{2}{3}}$.

4504. The parabola $y = x^2 - 2ax + a^2 + b$, drawn on (x, y) axes, is rotated by a right angle clockwise around its vertex. Find, in a similar form, the equation of its image.

4505. Determine the value of $\displaystyle\lim_{x\to 1} \frac{x^5 - 1}{x^5 - x}$.

4506. Sketch $y = \cos^5 x$.

4507. A cylindrical machine part has three components mounted on it, with masses in the ratio $1 : 2 : 4$. The components are attached at the corners of an equilateral triangle. The axle is smooth, and the part is free to spin in the vertical plane shown. It is in equilibrium, with the radius to the $4m$ mass making an angle θ with the downward vertical.



Find the exact value of $\tan\theta$.

4508. Show that the shortest path between the graphs $y = e^x$ and $y = \ln x$ has length $\sqrt{2}$.

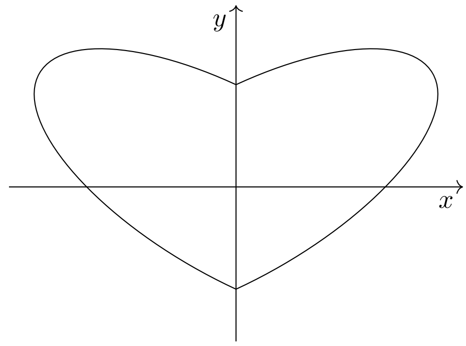
4509. Show that $\displaystyle\int_0^2 \frac{225x^2 + 30x + 1}{15x^2 + 31x + 2}\,dx = 30 - 29\ln 2$.

4510. From first principles, prove that

$$\frac{d}{dx}\left(\frac{2}{1 - \sqrt{x}}\right) \equiv \frac{1}{\sqrt{x}\left(1 - \sqrt{x}\right)^2}.$$

4511. The following curve has equation

$$\left(y - |x|\right)^2 + y^2 = 1.$$



Find the coordinates of the points where

(a) the tangent is parallel to the x axis,

(b) the tangent is parallel to the y axis.

4512. A particle has position $\mathbf{r} = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j} + t\,\mathbf{k}$.

(a) Show that, for $t \geq 0$, the distance from the origin increases at a constant rate.

(b) By sketching a plan (x, y) view and side (x, z) view of the trajectory, describe the motion.

4513. Assuming the double-angle identities, prove the trigonometric half-angle identity

$$\tan\tfrac{1}{2}\theta \equiv \frac{1 - \cos\theta}{\sin\theta}.$$

4514. A family of circles is defined, for $p \in \mathbb{R}$, by

$$\left(x - p\right)^2 + \left(y - |p|\right)^2 = 1.$$

On a sketch, shade the points of the (x, y) plane which lie on at least one of the circles.

4515. Simultaneous equations are given as

$$\ln x + \ln y = 2\ln z,$$
$$x + y = 2z.$$

Show that, if both equations hold, then $x = y$.

4516. This question concerns the integral of $\arctan x$.

(a) By writing $x = \tan y$, show that

$$\frac{d}{dx}\left(\arctan x\right) \equiv \frac{1}{1 + x^2}.$$

(b) Using integration by parts, show that

$$\int \arctan x\,dx = x\arctan x - \tfrac{1}{2}\ln\left(1 + x^2\right) + c.$$

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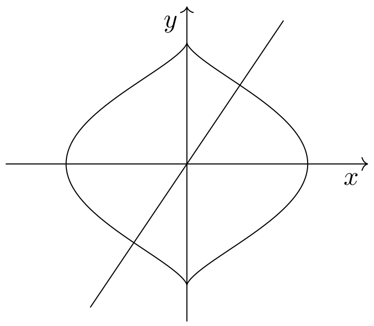
4517. Prove that the origin is the only point at which a tangent to $y = x^3 - x$ can be drawn that does not re-intersect the curve.

4518. The diagram below shows the curve

$$x^2 = \left(1 - y^2\right)^3.$$

Also shown is a normal to the curve which passes through the origin.



Find, to 3sf, the gradient of the normal.

4519. Four beads of different colours are threaded onto a bracelet. Assuming that a rotation of the bracelet does not count as a new arrangement, find the number of different ways of arranging the beads.

4520. A non-linear, non-separable DE is given as

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = 12.$$

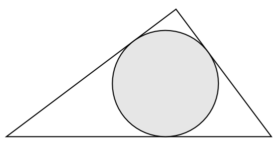
Show that any polynomial solution curve $y = \mathrm{f}(x)$ must be linear.

4521. An equation is given as

$$\cos^3\theta\sin\theta - 2\sin\theta + \tfrac{1}{2}\cos^3\theta = 1.$$

Solve for $\theta \in [-\pi, \pi]$.

4522. Find the radius of the largest circle that can be contained within a $(3, 4, 5)$ right-angled triangle.



4523. Curve C has equation $y^3 = 1 + x^3$.

(a) Show that, for $x, y \to \pm\infty$, C is asymptotic to the line $y = x$.

(b) Show that C is parallel to the axes at $(0, 1)$ and $(1, 0)$.

(c) Hence, sketch C.

4524. A monic quartic function f is invertible over each of the domains $\{x : x \leq p\}$ and $\{x : x \geq p\}$, where $\mathrm{f}(p) > 0$. The equation $\mathrm{f}(x) = 0$ has n roots.

Determine all possible values of n.

4525. In a circus, a light, inextensible rope is passed over a smooth, fixed pulley. A small platform of mass $2m$ is attached to each end of the rope. On each platform stands a monkey of mass $2m$. Each of the monkeys is carrying a large banana of mass m. Initially, the system is at rest, with the monkeys next to one another, at the same vertical height.

Then... kaboom! One monkey snatches the other's banana, and immediately accelerates downwards out of reach.

One second later, unwilling to accept the loss of its banana, the victim drops from its rising platform and dives in pursuit.

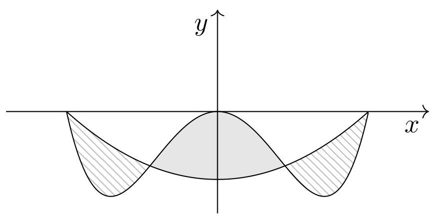
Show that the thief is caught two seconds later.

4526. Prove that $\dfrac{d^2}{dx^2}(x^x) \equiv x^x(\ln x + 1)^2 + x^{x-1}$.

4527. The diagram below shows, for $x \in [-1, 1]$ and some constant $k \in (0, 1)$, the curves

$$y = x^4 - x^2,$$
$$y = k(x^2 - 1).$$

They intersect at $x = \pm 1$, and enclose three areas between those x values. You are given that the area of the central region is the same as both of the others combined.



Show that $k = \frac{1}{5}$.

4528. Sets of p white counters and q black counters are to be placed on a board of n squares, with no more than one counter per square. Prove that, if this can be done, then the number of different ways of doing it is given by

$$\frac{n!}{p! \times q! \times (n - p - q)!}.$$

4529. A student proposes partial fractions in the form

$$\frac{x^2 + 1}{(x-3)(x-5)} \equiv \frac{A}{x-3} + \frac{B}{x-5}.$$

Explain why this will not work, and perform the preliminary step required.

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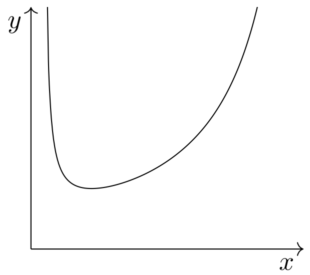
4530. A cubic graph $y = \mathrm{f}(x)$ intersects the x axis at three distinct points $x = p, q, r$.

Find the x coordinate of the point of inflection of $y = \mathrm{f}(x)$, giving your answer in terms of p, q, r.

4531. A part of the graph $y = \sec(\ln x)$ is shown below. The interval $\{x \in \mathbb{R} : 1 \leq x \leq 2\}$ is part of the restricted domain shown in this graph.



Prove that $\displaystyle\int_1^2 \sec(\ln x)\,dx > 1$.

4532. Two independent experiments are modelled with binomial distributions. These are

$$X \sim B(20, 0.1),$$
$$Y \sim B(30, 0.1).$$

(a) Write down the distribution of probabilities for the sum of the outcomes $X + Y$.

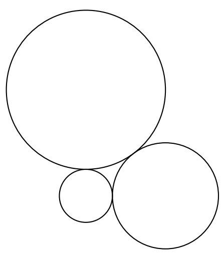
(b) By calculating and comparing probabilities, verify explicitly your answer to part (a), in the case where the sum of the outcomes is 2.

4533. A truck, of mass m_1, is pulling a trailer, of mass m_2, along flat horizontal ground, by means of a light, rigid tow-bar. Resistances R_1 and R_2 act on the truck and trailer respectively. The driver applies the brakes, and a braking force F acts on the truck. Prove that, if

$$F > \frac{m_1}{m_2}R_2 - R_1,$$

then the force in the tow-bar will be a thrust.

4534. Show that the set of tangential circles below, radii 1, 2 and 3 units, can be placed inside a rectangle of area 60 square units.



4535. Prove the following result:

$$\left(\sum_{r=1}^{n} r\right)^2 + (n+1)^3 = \left(\sum_{r=1}^{n+1} r\right)^2.$$

4536. Prove that $\displaystyle\sum_{r=0}^{n} {}^n\mathrm{C}_r = 2^n$.

4537. The depth, d cm, of a small metal ball, when it is dropped through a viscous liquid, is modelled with the formula

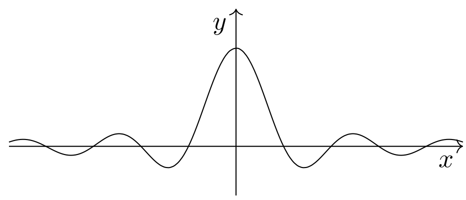
$$d = \frac{t^2}{10t + 25}.$$

(a) Show that the model describes the metal ball as being released from rest at the surface of the liquid.

(b) The long-term behaviour is asymptotic to a linear descent $d = at + b$, where a, b are real constants. Determine the values of a and b.

4538. The *sinc function* is defined as

$$\operatorname{sinc} x = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$



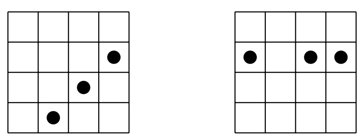
(a) Show that the function is continuous at $x = 0$.

(b) Find, to 3sf, the range of the function.

4539. An equation $\mathrm{f}(x) = 0$, with roots $\alpha_1 < \alpha_2 < ...$, is rearranged to $x = \mathrm{g}(x)$, where g is an increasing polynomial function. The iteration $x_{n+1} = \mathrm{g}(x_n)$ is then set up, and a starting value x_0 chosen in an interval (α_k, α_{k+1}) between two consecutive roots.

Prove that every starting point x_0 in this interval converges to the same root.

4540. A set of $n - 1$ counters is to be placed on an $n \times n$ grid, with a maximum of one per grid square, such that they are collinear.



Prove that the number of different ways of doing this is $2n^2 + 2n + 4$.

4541. Solve $\operatorname{cosec} x + \sec x = \sqrt{8}$ for $x \in [0, \pi)$.

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4542. In this question, use the following result.

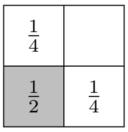
When a particle is projected from height h above the ground, with speed u, at angle of projection θ, then the equation of its trajectory in the (x, y) plane of its motion is

$$y = h + x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta).$$

A firework explodes h metres above the ground. Tiny fragments are propelled in every direction at speed u. They are modelled as projectiles. Show that fragments will land on a region of radius

$$r = \frac{u\sqrt{u^2 + 2gh}}{g}.$$

4543. A computer programmer is modelling a random walk on a 2×2 grid. A shaded square marks the position. At each iteration, the probabilities for the new location of the shaded square, relative to the current position, are as shown below.



Find the probability that, after four iterations, the shaded square is where it began.

4544. Three identical Christmas baubles, modelled as smooth spheres of mass m and radius r, are hung from the same fixed point. Each is hung by a string of length r. The baubles hang symmetrically in equilibrium. Show that each tension is

$$T = \tfrac{\sqrt{6}}{2}mg.$$

4545. In this question, do not use a calculator.

A quartic equation is given as

$$x^4 - 7x^3 + 18x^2 - 20x + 8 = 0.$$

You are told that this equation has a triple root at $x = a$, where $a \in \mathbb{Z}^+$. Determine the solution of the equation.

4546. A family of curves C_n is defined by the equation $x^n + y^n = 1$, for $n \in \mathbb{N}$. Show that

(a) for odd n, the curve C_n is unbounded,

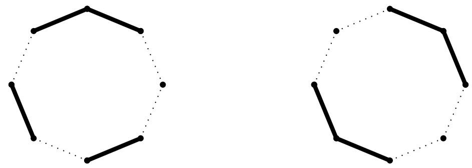
(b) for even n, the curve C_n is bounded.

Note: in this context, *bounded* means that there is an upper bound on distance from the origin. In other words, the curve doesn't extend to infinity.

4547. Express $\cos 3\theta$ in terms of $\cos\theta$.

4548. The area of the region enclosed by $y = \ln x$, the axes and the line $y = k$ is $e^2 - 1$. Determine the value of k.

4549. On a regular octagon, four of the edges are selected at random. Two examples are shown.



Find the probability that, as in the diagram on the right, the four selected edges form two pairs of opposites.

4550. Solve, for $x, y \in [0, \pi/2)$,

$$4\sqrt{3}\sin x + 2\cos y = 6 + \sqrt{2},$$
$$2\cos x - \sqrt{2}\sin y = 0.$$

4551. Sketch the regions of the (x, y) plane that satisfy the inequality $4x^2 - y^2 \geq 0$.

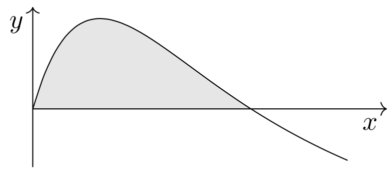
4552. Solve $27^x + 3^x = 9^{x+1} + 9$.

4553. A smooth, taut, light rope has a transverse force P Newtons applied to its midpoint, producing two sections of rope at an obtuse angle θ to each other. Show that the tension in the rope satisfies

$$T = \frac{P}{\sqrt{2 + 2\cos\theta)}}.$$

4554. A cubic graph $y = \text{f}(x)$ has two distinct stationary points. Prove that the stationary points and its point of inflection are collinear.

4555. The diagram shows the graph $y = xe^{2-x} - x$.



Find the exact area of the shaded region.

4556. Three currencies are fluctuating in value. They are modelled with three variables x, y, z, changing in time t. The variables are related by $\frac{dy}{dx} = 5$ and $\frac{dy}{dz} = -2$, and the rate of change of x is 4.

(a) Find the rate of change of z.

(b) The variables are all reset to 100. Assuming the same derivatives, determine a formula for the sum $x + y + z$ in terms of elapsed time t.

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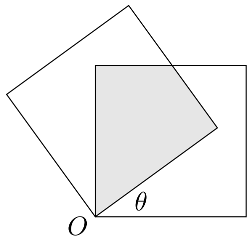
4557. In Pascal's triangle, the r^{th} entry in the nth row is given by ${}^n\mathrm{C}_r$. Prove that, if n is prime, then every entry in row n is either 1 or a multiple of n.

4558. Two unit squares are drawn with a vertex at O. One square is rotated by angle θ anticlockwise about O.



(a) Show that, for $\theta \in [0°, 90°]$, the area of the shaded region is given by

$$A = \frac{1 - \tan \frac{1}{2}\theta}{1 + \tan \frac{1}{2}\theta}.$$

(b) Sketch A against θ over $[0°, 360°]$.

4559. Curve C has the following equation:

$$y = 4e^{2-x} - e^{4-2x}.$$

Point (a, b) lies a distance of 1 unit from curve C.

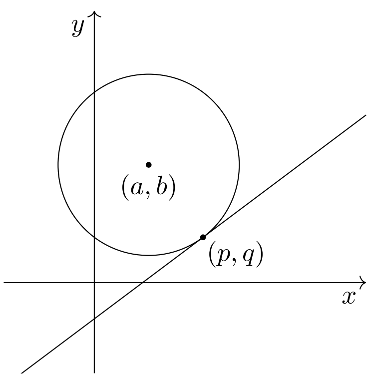
Find all possible values of a and all possible values of b. Give your answers as a set for each: there is no need to connect the answers.

4560. An iteration is defined, for constant $k \in \mathbb{N}$, by

$$x_{n+1} = \sum_{r=1}^{2k} x_n^r.$$

The iteration has a trivial fixed point at $x_n = 0$. Prove that it has no other fixed points.

4561. The circle $(x - a)^2 + (y - b)^2 = r^2$ has a tangent drawn to it at the point (p, q).



Prove that this tangent has equation

$$(x - p)(p - a) + (y - q)(q - b) = 0.$$

4562. A curve is given, for positive constants a, b, by

$$ax + by = b^2x^2 - 2abxy + a^2y^2.$$

Show that this curve is a parabola, and sketch it.

4563. A function has instruction $\mathrm{f}(x) = \dfrac{1}{x+1}$.

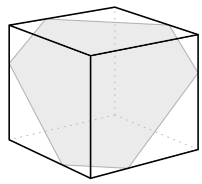
(a) Write down the largest real domain over which f can be defined.

(b) Find the largest real domain over which f^2 can be defined.

4564. Events A_i have probabilities given by $\mathbb{P}(A_i) = \frac{1}{2}^i$, for $i \in \{1, 2, 3\}$. The three events are independent of one another. Find the probability of at least two of the events occurring.

4565. Show that $\displaystyle\int_0^\pi \cos^2 x \, dx = \int_0^\pi \cos^2 2x \, dx$.

4566. The diagram shows a cube of unit side length, with a hexagonal cross-section. The hexagonal cross-section is normal to a space diagonal of the cube.



(a) Explain why the side lengths of the hexagon must be (a, b, a, b, a, b), for some $a, b \in \mathbb{R}$.

(b) Show that a and b may be expressed as

$$a = \sqrt{2}x, \quad b = \sqrt{2}(1 - x).$$

(c) Hence, or otherwise, prove that the area of such a hexagonal cross-section is maximised when it is regular.

4567. An iteration is defined by

$$x_{n+1} = x_n^3 - x_n - 4.$$

Show that, if this is to diverge to positive infinity, then the starting value x_0 must satisfy $x_0 > 2$.

4568. Variables x and y satisfy the relationship

$$\frac{dy}{dx} = x \cos x \operatorname{cosec} y.$$

Show that $x \sin x + \cos x + \cos y$ is constant.

4569. By completing the square and using a suitable trigonometric substitution, determine

$$\int \frac{1}{x^2 + 2x + 2} \, dx.$$

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4570. A particle oscillates in a magnetic field, until the field is switched off. At time t, the particle is at $x = \sin t$, $y = 4.9$ metres, where x is horizontal and y vertical position. After the field is switched off, the particle becomes a projectile.

Find all possible landing points on the x axis.

4571. In the game paper-scissors-stone, each contestant chooses one of the three. Paper beats stone, which beats scissors, which beats paper.

Assuming that all outcomes are equally likely, and that choices are independent of one another, find the probability that

(a) the first three rounds are drawn,
(b) in a *first to two wins* competition, at least five rounds are required.

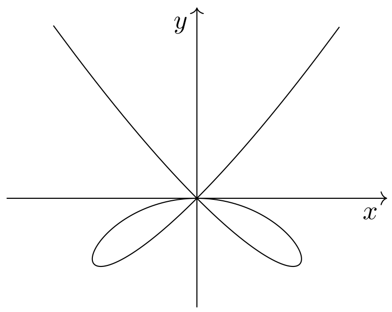
4572. Sketch and shade the regions of the (x, y) plane which satisfy the inequality

$$-x^2y + x^2 + y^3 - y^2 > 0.$$

4573. A parametric graph is defined, for $t \in \mathbb{R}$, by

$$x = t^3 - t,$$
$$y = t^4 - t^2.$$

The graph is shown below:



(a) Find the three t values at the origin.
(b) Show that the area of each loop is $\frac{4}{105}$.

4574. Prove that the product of four consecutive integers is one less than a perfect square.

4575. An equation is defined, for constants p, q, r, as

$$px^5 + qx^3 + rx = 0.$$

You are given that this equation has exactly three real roots. State, with a reason, whether each of the following equations can be guaranteed to have exactly three real roots:

(a) $px^6 + qx^4 + rx^2 = 0$,
(b) $px^{10} + qx^6 + rx^2 = 0$,
(c) $px^{15} + qx^9 + rx^3 = 0$.

4576. A differential equation is given as

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

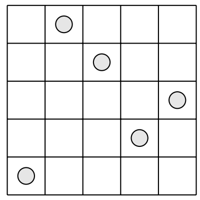
(a) Verify that $y = e^x$ is a solution.
(b) A second solution is proposed, in the form $y = \text{f}(x)e^x$ for some function f. Show that

$$\text{f}''(x) = 0.$$

(c) Hence, prove that the general solution of the original differential equation is

$$y = (Ax + B)e^x.$$

4577. On an n-by-n grid, n identical counters are placed at random, on distinct squares. With probability p, no two counters occupy the same row or column.



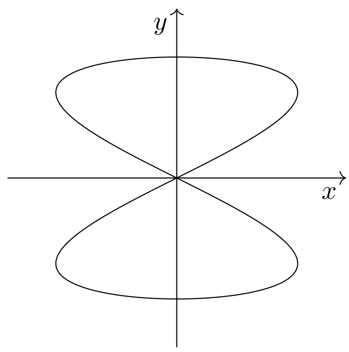
Prove that $p = \dfrac{(n^2 - n)!\big((n-1)!\big)^2}{(n^2 - 1)!}$.

4578. Factorise $2x^2 + xy - 3xz - y^2 - 6yz - 5z^2$.

4579. Show that $\sin y + \cos(xy) = 1$ has infinitely many stationary points on the y axis.

4580. Find the probability that four consecutive rolls of a die yield a strictly increasing sequence.

4581. A parametric shape known as a *Lissajous curve* is defined by $x = \sin 2t$, $y = \sin t$, where $-\pi < t \leq \pi$.



Determine the acute angle between the two parts of the curve that cross at the origin.

4582. Find the following integral, where $n \in \{2, 3, 4, ...\}$.

$$\int \frac{1}{x(\ln x)^n}\, dx.$$

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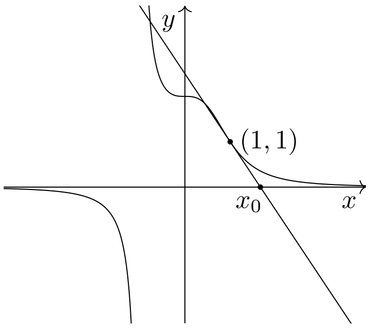
4583. The following equation, in which $n \in \mathbb{Z}$, defines a family of curves, all of which pass through $(1, 1)$:

$$y = \frac{2}{1 + x^n}.$$

On such a curve, a tangent is drawn at $(1, 1)$. This tangent crosses the x axis at x_0. The diagram shows an example of this, with $n = 3$:



Find the set of values of n for which $x_0 \geq 0$.

4584. The following differential equation, in which f is a function, is known as *Clairault's equation*:

$$y = x\frac{dy}{dx} + \mathrm{f}\left(\frac{dy}{dx}\right).$$

(a) Show that either $\dfrac{d^2y}{dx^2} = 0$ or $x + \mathrm{f}'\left(\dfrac{dy}{dx}\right) = 0$.

(b) For the first case, give the general solution.

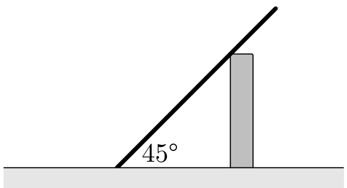
4585. The origin, $(\sin 2a, \cos 2a)$ and $(2\cos a, 2\sin a)$ are collinear. Find all possible values of $a \in [0, \pi)$.

4586. A function is defined for all real numbers by

$$\mathrm{g}(x) = (x + 1)(x + 3).$$

Show that $\mathrm{g}\big(|x|\big) - 2 = 0$ has no roots.

4587. A ladder of length $2l$ is placed against a low wall of height l, inclined at 45° to the horizontal. The ground is modelled as rough, with coefficient of friction μ, and the wall is modelled as smooth.



Show that, for equilibrium, $\mu \geq \dfrac{1 + 2\sqrt{2}}{7}$.

4588. Prove the following identity:

$$\sin^4 x \equiv \frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x}.$$

4589. On Cartesian axes, three straight lines are drawn:

$$y = ax + b,$$
$$y = cx + d,$$
$$y = ex + f.$$

(a) Show that, if $a = c = e$, then the lines either have no points in common or infinitely many.

(b) Show that, if the lines are concurrent, then

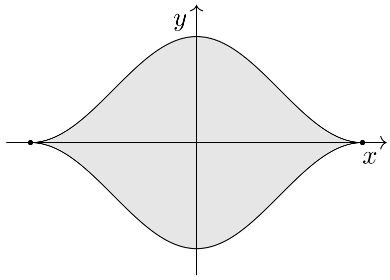
$$ad - bc = e(d - b) + \mathrm{f}(a - c).$$

4590. Solve the equation $e^x + e^{1-x} = e + 1$.

4591. An (x, y) inequality is given as

$$y^2 \leq 2 + 2\cos x - \sin^2 x.$$

The shaded region below represents the (x, y) points, over a domain of the form $[-k, k]$, which satisfy the inequality:



Find the exact area of the shaded region.

4592. Show that the derivative of $y = \arcsin x$ is

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

4593. A function has instruction

$$\mathrm{f}(x) = \ln x + 1.$$

The Newton-Raphson iteration is set up, using the function above and starting value x_0. Find the set of values of x_0 for which x_1 is well defined, but x_2 is not well defined.

4594. Show that $\displaystyle\int_0^{2\pi} (1 - \cos t)^2 \, dt = 3\pi$.

4595. A partial sum formula for a particular sequence is

$$\mathrm{S}(n) = \left(\frac{n^2 + n}{2}\right)^2.$$

(a) Prove that $\mathrm{S}(n) + (n + 1)^3 = \mathrm{S}(n + 1)$.

(b) Hence, write down the sequence of which $\mathrm{S}(n)$ is the partial sum formula.

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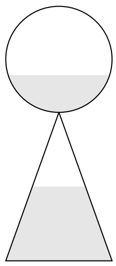
4596. In this question, you should use the fact that the *volume of a spherical cap* is given by $V = \frac{2}{3}\pi r^2 h$, where h is the height of the cap and r the radius of the sphere.

An hourglass consists of a fixed volume of water dripping from a cone to a sphere and vice versa. The cone and the sphere have the same radius r, and the water exactly fills each.



Show that, when the hourglass is oriented as in the diagram, the depth H in the cone and the depth h in the sphere satisfy

$$\left(4r - H\right)^3 = 32r^2 h.$$

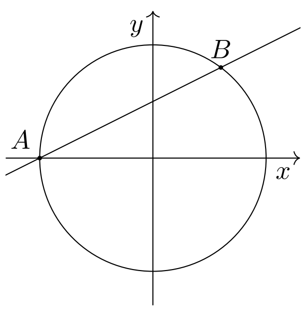
4597. Solve the inequality $2x^5 - 3x^3 + x \geq 0$, giving your answer exactly in set notation.

4598. In an industrial process, the excess quantities of two chemicals oscillate sinusoidally:

$$E_1 = \sin 2t,$$
$$E_2 = k\cos t, \text{ for some constant } k.$$

Show that, over a long period, the fraction of the time for which $E_1 > E_2$ is independent of k.

4599. A unit circle is drawn, together with a straight line $L_m : y = m(x + 1)$, for $m > 0$.



(a) Show that line and circle intersect at

$$A : x = -1,$$
$$B : x = \frac{1 - m^2}{1 + m^2}.$$

(b) Find the equation of the normal to L_m that passes through point B.

(c) Hence, prove that the angle in a semicircle is a right angle.

4600. An economist analysing a crude oil market models supply S and demand D, in millions of barrels, at time t in months, with the following equations:

$$\frac{dS}{dt} = 0.015D - 0.037S,$$
$$\frac{dD}{dt} = 0.027D + 0.021S.$$

At $t = 0$, $S = 57.1$ and $D = 51.3$.

(a) Find the initial rates of change of S and D.

(b) Assume, as a first approximation to the model above, that these initial rates remain constant. Find the predicted time at which supply and demand equalise.

(c) Assume instead, as a better approximation to the model, that the rates remain constant for month-long intervals. Find the predicted time at which supply and demand equalise.

— End of 46th Hundred —

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